TRANSIENT ANALYSIS OF AN AIR CONDITIONER AS A PLANT WITH DISTRIBUTED PARAMETERS

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A procedure is shown for the transient analysis of an air conditioner with plane-parallel packing and forward flow, when the dynamic performance is described by transcendental transfer functions.

Refrigeration systems and apparatus are usually analyzed for steady-state performance. Such an analysis makes it possible to determine their coefficients of heat and mass transfer. Heat treatment plants for food products operate always under unsteady conditions. In view of this, it becomes necessary to analyze transient performance modes. On the basis of such an analysis, it appears feasible to develop a rational system of automatic regulation, to define its performance requirements, and to design an overall efficient apparatus. By analyzing the effect which a variation of structural and thermophysical parameters has on the air-conditioner response, one can predict its performance and design an apparatus with the required dynamic characteristics.

It is well known that the transient performance can be described by transfer functions defining the response to respective perturbation and regulation signals, and we will here derive these functions as follows.

1. We write the differential equations of heat balance and material balance in the apparatus. The heat balance in the mass of humid air along element ΔH (of packing height) is

$$dQ_\alpha dH = \left(-bl\omega_a\gamma_a c_a n - bl\omega_a\gamma_a c_v n\frac{d\Delta d}{1000}\right)dt_a dH \quad (1)$$

and represents the amount of heat transmitted from the air to the liquid while the air temperature changes.

Since the quantity $(d\Delta d/1000)c_n$ is negligibly small as compared with c_a, hence the second term in Eq. (1) may be omitted. In view of what has been said here and because $dQ_\alpha = \alpha_a(t_a - t_w)l2ndH$, Eq. (1) can, after simple transformations, be written in partial derivatives

$$-blc_a n\gamma_a\frac{\partial t_a}{\partial\tau} - G_a c_a\frac{\partial t_a}{\partial H} = \alpha_a l2n(t_a - t_w). \quad (2)$$

The total amount of heat, sensible and latent, received by the dew collecting liquid mass is

$$G_w c_w dt_w = \left[\alpha_a l2n(t_a - t_w) + \beta l2n(p_a - p_n)\frac{r}{1000}\right]dH. \quad (3)$$

Having used the $P_{sat} = f(t)$ curve plotted from steam tables, we now replace it by the following relation valid for small temperature drops

$$p_H \cong \bar{n}(t_a - \varepsilon), \quad (4)$$

where n and ε are constant within any given temperature range.

Considering that $p_a = B - p_v$ and substituting $p_v = \varphi p_{sat}$ in (4), we obtain

$$p_a \cong -(\varphi + 1)\bar{n}t_a \quad (5)$$

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With expression (5) taken into account, Eq. (3) is transformed into a partial differential equation:

$$2lH\gamma_w nc_w \frac{\partial t_w}{\partial \tau} + G_w c_w \frac{\partial t_w}{\partial H} = (\alpha_a l2n - rm)\, t_a - \alpha_a l2nt_w, \tag{6}$$

where

$$\bar{m} = \frac{(\varphi + 1)\, 2n\beta l\bar{n}}{1000}.$$

2. Having performed the necessary transformations and having linearized the nonlinear relations $\alpha_a = \alpha_a(G_a, G_w)$, $\beta = \beta(G_a, G_w)$ in the preceding paragraph for solution by the method of small perturbations with zero initial and boundary conditions, we now perform Laplace transformations with respect to the coordinates (time and height H, the latter characterizing the distributiveness of parameters in the heat exchanger):

$$\begin{aligned} &(-s - A_1(p))\, t_a(s, p) = -a_2 t_w(s, p) + \frac{1}{s} a_3 G_a(0, p) + t_{in}(0, p), \\ &(s + B_1(p))\, t_w(s, p) = b_3 t_a(s, p) + \frac{1}{s} b_4 G_a(0, p) + t_{wH}(0, p). \end{aligned} \tag{7}$$

3. We then solve this system of equations (see preceding paragraph) in operator form and determine the roots $s_1(p)$, $s_2(p)$.

4. Performing the inverse Laplace transformation with respect to coordinate H (packing height), we obtain the transfer functions for an air conditioner with plane-parallel packing and forward flow:

$$W(H, p) = \frac{t_a(H, p)}{t_{in}(0, p)} = \frac{1}{s_2(p) - s_1(p)} \left[(-s_1(p) + B_1(p))\, e^{-s_1(p)H} + (s_2(p) - B_1(p))\, e^{-s_2(p)H}\right], \tag{8}$$

where

$$s_{1,2}(p) = -\frac{B_1(p) + A_1(p)}{2} \pm \sqrt{\left(\frac{B_1(p) + A_1(p)}{2}\right)^2 - (A_1(p) B_1(p) - a_2 b_2)};$$

$$B_1(p) = b_1 p + b_2; \quad A_1(p) = a_1 p + a_2;$$

where $s_{1,2}(p)$ are the roots of the system of equations describing the heat transfer in the air conditioner.

Known methods of solving Eqs. (1) require an approximation of transcendental functions [1, 2]. A drawback of such an approach is that one cannot estimate the order of the approximating function necessary for the desired accuracy of the solution.

Frequency methods, which have been widely used now for the analysis of automatic regulation systems, make it possible to plot the transient response of a plant to various perturbation and regulation signals without the need to approximate the transfer function (8).

This article, therefore, will deal with the procedure for plotting the transient response of a plant described by a transfer function like (8) when the input signal is a step function or an arbitrary time function.

In order to obtain the amplitude-phase characteristics of a plant, we substitute $p = j\omega$ in the transfer function (8) so that $s_1(p)$ and $s_2(p)$ become

$$s_{1,2}(j\omega) = \frac{b_1 + a_1}{2} j\omega + \frac{b_2 + a_2}{2} \pm \frac{1}{2}\sqrt{-(b_1 - a_1)^2 \omega^2 + 2[(a_1 - b_1)\, a_2 + (b_1 - a_1)\, b_2]\, j\omega + [(b_2 + a_2)^2 + 4a_2 b_3]}. \tag{9}$$

The expression under the square-root sign in (9) will now be transformed into

$$\sqrt{r(\cos\varphi + j\sin\varphi)} = \sqrt{r}\left(1 + j\frac{\sin\varphi}{1 + \cos\varphi}\right) \cdot \sqrt{\frac{1 + \cos\varphi}{2}}, \tag{10}$$

where

$$r = \sqrt{(c_2 - c_3\omega^2)^2 + c_1^2\omega^2}; \quad \cos\varphi = \frac{c_2 - c_3\omega^2}{\sqrt{(c_2 - c_3\omega^2)^2 + c_1^2\omega^2}};$$

$$\sin\varphi = \frac{c_1\omega}{\sqrt{(c_2 - c_2\omega^2)^2 + c_1^2\omega^2}}; \quad c_1 = 2\left[(a_1 - b_1)a_2 + (b_1 - a_1)b_2\right];$$

$$c_2 = (b_2 - a_2)^2 + 4a_2b_3; \quad c_3 = (b_1 - a_1)^2;$$

$$a_1 = \frac{bl\gamma_a n}{G_a}; \quad b_1 = \frac{2lH\gamma_w n}{G_w}; \quad b_2 = \frac{\alpha_{wo}2nl}{G_w c_w};$$

$$a_2 = \frac{\alpha_{wo}2nl}{G_a c_a}; \quad b_3 = \frac{10^3\cdot\alpha_{wo}2nl - r(1+\varphi)\,2\overline{nl}n\beta}{10^3\cdot G_w c_w}.$$

According to (10), the roots can be expressed as

$$s_1(j\omega) = \mathrm{Re}_1(\omega) + j\,\mathrm{Im}_1(\omega),$$
$$s_2(j\omega) = \mathrm{Re}_2(\omega) + j\,\mathrm{Im}_2(\omega). \tag{11}$$

After inserting the values (11) into the transfer function (8) and using the Euler formula

$$e^{-j\mathrm{Im}_2(\omega)H} = \cos\mathrm{Im}_2(\omega)H - j\sin\mathrm{Im}_2(\omega)H,$$
$$e^{-j\mathrm{Im}_1(\omega)H} = \cos\mathrm{Im}_1(\omega)H - j\sin\mathrm{Im}_1(\omega)H, \tag{12}$$

we will transform the transfer function (8) so as to separate then the real and the imaginary component

$$W(H, j\omega) = P(H, \omega) + jQ(H, \omega), \tag{13}$$

where

$$\left.\begin{aligned} P(H, \omega) &= \frac{V_1(H, \omega)V_3(\omega) + V_2(H, \omega)\cdot V_4(\omega)}{V_3^2(\omega) + V_4^2(\omega)} \\ Q(H, \omega) &= \frac{V_2(H, \omega)V_3(\omega) - V_1(\omega, H)\cdot V_4(\omega)}{V_3^2(\omega) + V_4^2(\omega)} \end{aligned}\right\}, \tag{14}$$

$$V_1(H, \omega) = \left[(b_2 - \mathrm{Re}_1(\omega))\cos\mathrm{Im}_1(\omega)H + (b_1\omega - \mathrm{Im}_1(\omega))\sin\mathrm{Im}_1(\omega)H\right]$$
$$\times e^{-\mathrm{Re}_1(\omega)H} + \left[(\mathrm{Re}_2(\omega) - b_2)\cos\mathrm{Im}_2(\omega)H + (\mathrm{Im}_2(\omega) - b_1\omega)\sin\mathrm{Im}_2(\omega)H\right]e^{-\mathrm{Re}_2(\omega)H}, \tag{15}$$

$$V_2(H, \omega) = \left[(b_1\omega - \mathrm{Im}_1(\omega))\cos\mathrm{Im}_1(\omega)H - (b_2 - \mathrm{Re}_1(\omega))\sin\mathrm{Im}_1(\omega)H\right]$$
$$\times e^{-\mathrm{Re}_1(\omega)H} + \left[(\mathrm{Im}_2(\omega) - b_1\omega)\cos\mathrm{Im}_2(\omega)H - (\mathrm{Re}_2(\omega) - b_2)\sin\mathrm{Im}_2(\omega)H\right]e^{-\mathrm{Re}_2(\omega)H}, \tag{16}$$

$$V_4(\omega) = \mathrm{Im}_2(\omega) - \mathrm{Im}_1(\omega); \quad V_3(\omega) = \mathrm{Re}_2(\omega) - \mathrm{Re}_1(\omega). \tag{17}$$

In order to plot the transient response which will simulate the time-variation of the air temperature at the air-conditioner outlet $t_a(\tau)$ during a variation of the air temperature at the inlet $t_{in}(\tau)$, we use the relation between the transient function $t_a(\tau)$ and the real or the imaginary component of the frequency characteristic of the air-conditioner [3].

We consider here two cases.

a. If the perturbations at the air-conditioner inlet vary stepwise, then the transient function $t_a(\tau)$ is related to the real and the imaginary component of the frequency characteristic (11) as follows [3]:

$$t_a(\tau) = \frac{2}{\pi}\int_0^\infty \frac{P(H, \omega)}{\omega}\sin\tau\omega d\omega, \quad \tau > 0, \tag{18}$$

$$t_a(\tau) = P(0) + \frac{2}{\pi}\int_0^\infty \frac{Q(H, \omega)}{\omega}\sin\tau\omega d\omega, \quad \tau > 0, \tag{19}$$

where P(0) is the value of the real component at $\omega = 0$.

Using the concepts of an integral sine, introducing the dimensionless parameters

$$\varkappa = \frac{\omega_1}{\omega_0}, \quad \tau = \frac{\overline{\tau}}{\omega_0}$$

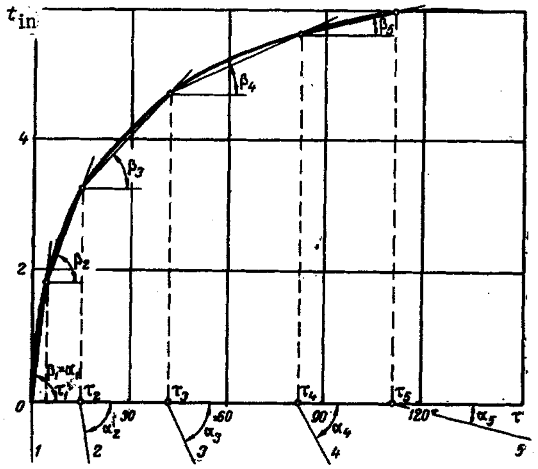

Fig. 1

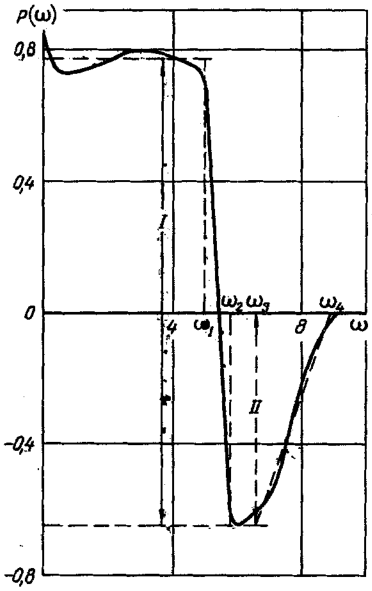

Fig. 2

Fig. 1. Input signal resolved into the sum of ramp signals: 1) $a_1\tau$; 2) $a_2(\tau-\tau_1)$; 3) $a_3(\tau-\tau_2)$; 4) $a_4(\tau-\tau_3)$; 5) $a_5(\tau-\tau_4)$; temperature t_{in} (°C), time τ (min).

Fig. 2. Real component of the frequency characteristic of an air conditioner, ω (1/min).

and letting $\omega_0 = 1$, we rewrite expression (18) as

$$h_{\varkappa_i}(\tau) = \frac{2}{\pi} P(0) \left\{\mathrm{Si}(\varkappa, \bar{\tau}) + \frac{1}{1-\varkappa_i}\left[\mathrm{Si}(\bar{\tau}) - \mathrm{Si}(\varkappa_i, \tau) + \frac{\cos\bar{\tau} - \cos\varkappa_i\bar{\tau}}{\bar{\tau}}\right]\right\}. \tag{20}$$

When $P_1(0) = 1$, expression (20) yields a unit transient function which has been tabulated in [3, 4]. With the aid of (14)-(17), we thus plot the real component of the frequency characteristic and approximate it by a series of polygon segments, the transient response corresponding to each segment found from the expression:

$$t_a(\tau)_i = h_{\varkappa_i}(\bar{\tau}) P_i(0). \tag{21}$$

Finally, the transient response of the plant to a step change in the input signal will be

$$t_a(\tau) = \sum_{i=1}^{i=n} h_{\varkappa_i}(\bar{\tau}) P_i(0), \tag{22}$$

with n denoting the number of polygon segments approximating the real component of the frequency characteristic (14) and $\varkappa$ denoting the slope of the respective polygon segments (which can vary from 0 to 1).

b. The input signal (air temperature at the inlet) varies not stepwise but arbitrarily in a quasi-exponential manner.

For this case we will consider the relation between an input signal varying along curve $t_{in}(\tau)$ (Fig. 1) and the outlet air temperature, the latter following in accordance with the transfer function (8). We will use here a piecewise-linear approximation of the input perturbation, representing the latter as the sum of linear signals [5]. The transient response $t_{al}(\tau)$ of a plant described by the transfer function (8) to a linear input signal $t_{in}(\tau) = \tau$ can, in fact, be represented as the sum of two components: a linearly increasing one $t_{al_1}(\tau)$ and a bounded one $t_{al_2}(\tau)$:

$$t_{al}(\tau) = t_{al_1}(\tau) - t_{al_2}(\tau), \tag{23}$$

where

$$t_{al_1}(\tau) = k\tau = P(0)\tau.$$

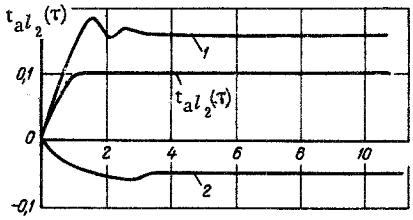

Fig. 3

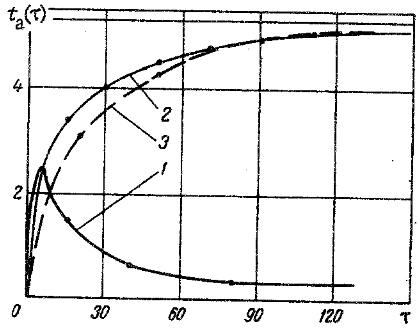

Fig. 4

Fig. 3. Plot of function $t_{al_2}(\tau)$: time τ (min).

Fig. 4. Plotted transient response of an air conditioner: 1) curve representing $\sum_{i=1}^{5} a_{i+1} t_{al_2}(\tau-\tau_i)$; 2) transient response determined analytically; 3) transient response measured in tests; temperature $t_a(\tau)$ (°C), time τ (min).

TABLE 1. Parameters of the Polygon Segments

No. of polygon segment	Value of the ordinates	Flat range of the pass band, 1/min	Total range of pass band, 1/min	Slope
I	$P_1(\omega, H)=1{,}43$	$\omega_1=5$	$\omega_{01}=5{,}8$	$\varkappa_1=0{,}86$
II	$-P_2(\omega, H)=0{,}65$	$\omega_2=6{,}6$	$\omega_{02}=8{,}9$	$\varkappa_2=0{,}74$

On the basis of a piecewise-linear approximation, an input signal of arbitrary form can be represented as the sum of two ramps with a definite delay between them:

$$t_{in}(\tau) = \sum_{i=0}^{n} a_{i+1}(\tau-\tau_i), \tag{24}$$

where

$$a_{i+1} = \operatorname{tg}\alpha_{i+1} = \operatorname{tg}\beta_{i+1} - \operatorname{tg}\beta_i \quad \text{(see Fig. 1)}$$

The transient response to an input signal (24), with (23) taken into account, can be expressed as [5]

$$t_a(\tau) = P(0)\tau - \sum_{i=0}^{n} a_{i+1} t_{al_2}(\tau-\tau_i). \tag{25}$$

Using the results in [5], we obtain a relation between the bounded component $t_{al_2}(\tau)$ and the real or the imaginary component of the frequency characteristic, whereupon we plot the transient response graphically.

Under actual conditions, the air temperature at an air-conditioner outlet is usually affected by several perturbing and regulating factors (change in the air flow rate, change in the inlet air temperature) which may occur simultaneously and in step or in any other arbitrary form. It is thus interesting to plot the $t_{al_2}(\tau)$ curve based on the real component of the frequency characteristic of the plant or, in this case, it is possible to use the same $P(H, \omega)$ characteristic for analyzing the air-conditioner response to a unit-step input (with the aid of $h_\varkappa$-function tables [3, 4]) as well as to an arbitrary input signal.

The real component of the frequency characteristic derived here we will also approximate by the sum of piecewise-linear frequency characteristics and on this basis, then, be able to write

$$t_{al_2}(\tau) = \sum_{i=1}^{n} l_i(\tau), \tag{26}$$

TABLE 2. Parameters of the Approximation to a Perturbation Signal

Parameter	Curves in Fig. 1				
	1	2	3	4	5
β_i	85°	72°	48°	27°	15°
a_i	a_1=11,43	a_2=7,25	a_3=1,96	a_4=1,45	a_5=0,241
τ_i	5 min	15 min	42 min	82 min	111 min

where $l_i(\tau)$ denotes the transient response to a linear frequency characteristic.

Introducing again the integral sine $\mathrm{Si}(\tau, \omega)$ and the integral cosine $\mathrm{Ci}(\tau, \omega)$ for a unit polygonal frequency characteristic ($P_1(0) = 1$, $\omega_0 = 1$) with $\varkappa = \omega_1/\omega_0$, we obtain from (26) [3]

$$l_\varkappa(\tau) = \frac{2}{\pi}\tau\left\{\frac{\pi}{2} - \mathrm{Si}(\varkappa\tau) - \frac{1}{1-\varkappa}\left[\mathrm{Si}(\tau) - \mathrm{Si}(\varkappa\tau) + \frac{\cos\tau - \cos(\varkappa\tau) - \mathrm{Ci}\,\tau - \mathrm{Ci}(\varkappa\tau) + \ln\varkappa}{\tau}\right]\right\}, \quad (27)$$

analogous to (20). Expression (27) is a function of $\varkappa$ and has been tabulated in [5].

In order that $l_i(\tau)$ which corresponds to a polygonal characteristic with parameters $P_i(0)$ can be found with the aid of these tables, it is necessary that

$$l_i(\tau) = l_\varkappa^{\mathrm{T}}(\tau)\,\frac{P_i(0)}{\omega}, \quad (28)$$

$$\tau = \frac{\tau^{\mathrm{T}}}{\omega_0}, \quad (29)$$

where $l_\varkappa^{\mathrm{T}}(\tau)$ are the ordinates of the tabulated function $l_\varkappa(\tau)$ which correspond to a determined value of $\varkappa$ and τ^{T} is the value of the argument given in Table 2.

The method shown here will now be illustrated on the transient response of an air-conditioner channel characterized by the transfer function (8). We will calculate the air-conditioner performance in the dry mode ($\varphi = 50\%$) with the following coefficients: $b_1 = 0.436$; $b_2 = 0.524$; $a_1 = 0.0024$; $a_2 = 0.304$; $b_3 = 0.528$. These coefficients depend on the air-conditioner geometry and on the thermophysical parameters of heat and mass transfer in the apparatus. The inlet air temperature varies according to the curve in Fig. 1.

With the aid of expressions (11), (10), and (13)-(17), we plot the real component of the frequency characteristic from $\omega = 0$ to $\omega = 9.1$/min (Fig. 2), which is then approximated by two polygon segments with parameters given in Table 1. The input perturbation (Fig. 1) is approximated by the sum of two ramp signals (Table 2).

For the case of a unit ramp input $t_{\mathrm{in}}(\tau) = \tau$ we then plot $l_1(\tau)$ and $l_2(\tau)$ (Fig. 3) corresponding to the respective polygon segments, using for this purpose the tables of $l_\varkappa$-functions [5] in accordance with expressions (22), (28), and (29). The resultant curve $t_{al_2}(\tau)$ represents the sought function (26).

Expressing the perturbation signal as the sum of five ramps with definite delays between them (Table 2)

$$t_{\mathrm{in}}(\tau) \cong a_1(\tau) + a_2(\tau - \tau_1) + a_3(\tau - \tau_2) + a_4(\tau - \tau_3) + a_5(\tau - \tau_4),$$

we plot the curve $\sum_{i=1}^{n} a_{i+1} t_{al_2}(\tau - \tau_i)$ (Fig. 4) with the aid of the previously plotted $t_{al_2}(\tau)$ curve (Fig. 3).

With this curve and according to (25), we find the transient response of the plant (Fig. 4) to a perturbation signal represented in Fig. 1.

In Fig. 4 is shown the transient response of an air conditioner, based on tests under the same conditions ($\varphi = 50\%$).

The transient response curves obtained here indicate that this method is suitable for the analysis of an air-conditioner response to various perturbing and regulating signals.

NOTATION

is the air temperature at the air-conditioner outlet;

t_{in}	is the air temperature at the air-conditioner inlet;
a_1, a_2, b_1, b_2, b_3	are the coefficients depending on the thermophysical properties of the air and of the liquid coolant as well as on the parameters of heat and mass transfer in the air conditioner;
φ	is the relative humidity of air at the air-conditioner inlet;
$Re_1(\omega)$, $Re_2(\omega)$, $P(H, \omega)$	are the real components of frequency characteristics;
$Im_1(\omega)$, $Im_2(\omega)$, $Q(H, \omega)$	are the imaginary components of frequency characteristics;
ω	is the frequency;
α, β	are the coefficients of heat and mass transfer, respectively;
$r = 597 + 0.55 t_w$	is the latent heat of water evaporation under the given operating temperatures;
l, H	are the packing width and height, respectively;
$\underline{n}$	is the number of packings;
$\bar{n}$	is the proportionality factor in the approximated relation between saturated-vapor pressure and air temperature;
γ_a, γ_w	are the densities of air and of liquid coolant, respectively;
G_a, G_w	are the flow rates of air and of liquid coolant, respectively;
c_a, c_w	are the specific heat of air and of liquid coolant, respectively.
$s = \partial/\partial H$, $p = \partial/\partial \tau$	are the differential operators with respect to coordinates H and τ, respectively.

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